Phase diagram of a generalized Winfree model

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We study the phase diagram of a generalized Winfree model. The modification is such that the coupling depends on the fraction of synchronized oscillators, a situation which has been noted in some experiments on coupled Josephson junctions and mechanical systems. We let the global coupling k be a function of the Kuramoto order parameter r through an exponent z such that $z=1$ corresponds to the standard Winfree model, $z<1$ strengthens the coupling at low r (low amount of synchronization), and at $z>1$, the coupling is weakened for low r. Using both analytical and numerical approaches, we find that z controls the size of the incoherent phase region and that one may make the incoherent behavior less typical by choosing $z<1$. We also find that the original Winfree model is a rather special case; indeed, the partial locked behavior disappears for $z>1$. At fixed k and varying γ, the stability boundary of the locked phase corresponds to a transition that is continuous for $z<1$ and first order for $z>1$. This change in the nature of the transition is in accordance with a previous study of a similarly modified Kuramoto model.

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I. INTRODUCTION

Forty years ago in a pioneering study Winfree [1] (see also [2]) introduced a mean-field model to describe the limit-cycle behavior of large populations of biological oscillators. He discovered that systems of oscillators with randomly distributed frequencies remain incoherent when the variance of the frequencies is reduced, until a certain threshold is reached. Subsequently the oscillators begin to synchronize spontaneously and become locked.

In its simplest form the model is defined by the set of equations $(i=1,\ldots,N, N\gg 1)$

$$\dot{\theta}_i(t) = \omega_i + \frac{k}{N}\sum_{j=1}^{N} P(\theta_j)R(\theta_i). \quad (1)$$

$\theta_i(t)$ is the phase of the ith oscillator; $\{\omega_i\}$ describes a set of natural frequencies taken randomly from a distribution $g(\omega)$. We shall assume below $g(\omega)=1/2\gamma$ for $\gamma\in[1-\gamma,1+\gamma]$ and $g(\omega)=0$ otherwise; $R(\theta_i)$ is the sensitivity function giving the response of the ith oscillator, and $P(\theta_j)$ is the influence function of the jth oscillator. A common choice is

$$R(\theta) = -\sin\theta, \quad P(\theta) = 1+\cos\theta. \quad (2)$$

Despite its historical merits the Winfree model has its own limitations. On the one side, it is complex enough not to admit a full analytical treatment. On the other side, it is not sufficiently sophisticated as to allow the treatment of realistic systems. Limitations of the former type were overcome by the work of Kuramoto [3] (for a review see [4]), who presented a model of oscillators related to the Winfree model (it is the weak-coupling limit of it) and analytically solvable in the mean-field approximation. Kuramoto's approach generated an intense theoretical work [5], also motivated by the fact that the phenomenon of mutual synchronization of coupled nonlinear oscillators is ubiquitous in nature, with applications to neural networks, networks of cardiac pacemaker cells, and populations of fireflies and crickets [2,6] as well as arrays of Josephson junctions [7].

It must be also mentioned that, in spite of the complexity of the Winfree model, its phase diagram was the object of investigation [8,9] and in the (k,γ) plane a rich structure was found. As to the versatility of the Winfree model it can be mentioned that it can describe different sets of pulse-coupled biological oscillators; see, e.g., [10–13]. In view of its relevance it might be interesting to look for extensions of the model that allow a different treatment of the couplings. Hopefully these extensions should enlarge the class of physical instances where the model can be usefully applied—for example, experimental setups like arrays of Josephson's junctions [14,15] or the crowd synchronization phenomenon on the Millenium Bridge [16].

The purpose of this work is to generalize the Winfree model to the case of a global coupling depending on the fraction of synchronized oscillators. In a recent work [17] a modification of the original Kuramoto model was presented. The authors of [17] noted that the natural control parameter of the Kuramoto model is a coupling strength, analogous to the k parameter in Eq. (1), independent of the number of oscillators that are locked in frequencies. They suggested to generalize the theoretical model by allowing a dependence of the coupling on the number of locked oscillators. The technical way to achieve this is to introduce a functional dependence on the Kuramoto order parameter r, which is defined by the equation

$$r(t)e^{i\psi(t)} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j(t)}. \quad (3)$$

Clearly in a locked or partially locked phase r does not vanish and its variability produces the desired functional depen-

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dence. As the Kuramoto model is the averaged system of the Winfree model, it is desirable to study the effect in this latter case. Therefore in this paper we study the effect of an r-dependent coupling in the Winfree model, along the lines of [8,9]. Our main result is that in some cases the modification produces an enlargement of the region, in parameter space, where locking or partial locking of the oscillators is possible. It is worth noting that another interesting extension of the Kuramoto model, where additional powers of the order parameter are introduced to account for the dependence of the form of the coupling on its magnitude, has been recently studied [18]. The plan of the paper is as follows. In the next section we introduce the generalized Winfree model, while in Sec. III we describe how we obtain the transition lines in the phase diagrams. Some conclusions are drawn in Sec. IV.

II. GENERALIZED WINFREE MODEL

We will consider the model defined by the set of equations

$$\dot{\theta}_i(t) = \omega_i + \frac{kr^{z-1}}{N}\sum_{j=1}^{N} P(\theta_j)R(\theta_i) \tag{4}$$

analogously to Eq. (1), with z a real parameter. It describes a set of coupled nonlinear oscillators, with coupling constant kr^{z-1}. For $z=1$ the model reduces to the Winfree model of Eq. (1).

For $N\to\infty$, the sum over all oscillators in Eq. (1) can be replaced by an integral, yielding the following equation for the velocity $v=\dot{\theta}$:

$$v(\omega,\theta,t) = \omega - \sigma(t)\sin\,\theta, \tag{5}$$

where

$$\sigma(t) = kr^{z-1}\int_0^{2\pi}\int_{1-\gamma}^{1+\gamma}(1+\cos\,\theta)p(\theta,t,\omega)g(\omega)d\omega d\theta. \tag{6}$$

Here $p(\omega,\theta,t)$ denotes the density of oscillators with phase θ at time t. It satisfies the continuity equation

$$\frac{\partial p}{\partial t} = -\frac{\partial(pv)}{\partial\theta} \tag{7}$$

and the normalization condition

$$\int_0^{2\pi} d\theta p(\theta,t,\omega) = 1. \tag{8}$$

for all ω and any time.

The phase diagram of the model with $z=1$ was studied by Ariaratnam and Strogatz [8]. They found a rich structure, comprising (i) *locked phase* (Lk), characterized by a common average frequency, i.e., a common value for the rotation number $\rho_i=\lim_{t\to\infty}\theta_i(t)/t$; (ii) *partial locking* (PL), characterized by macroscopic fractions of locked and unlocked oscillators; (iii) *incoherence* (In), where no macroscopic fraction of oscillators is locked to a common frequency; (iv) *death* (Dt), characterized by $\rho_i=0$ for any i; and (v) *partial death* (PD), where only a fraction the ρ_i vanishes. Moreover, they found several hybrid states that can be seen as different realizations of the partial locking phase [8].

We have numerically studied the model (4) with various values of the parameter z in the interval (0.5, 2.0) and various values of N, up to $N=1000$, starting from a random initial configuration. In general stability in the results is found after 500 time steps. Our results are qualitatively similar to those of the original Winfree model ($z=1$), but the boundaries between the different phases depend on the actual value of z. Our numerical analysis is in general confirmed by an analytical study; see the next section. There is one case, the boundary *locking-partial locking*, when analytical results are not available and the transition line must be evaluated only numerically.

The main outcome of the study is that the value of the parameter z controls the size, in the phase diagram, of the incoherent phase: one may make the incoherent behavior less typical by choosing $z<1$. The original Winfree model, corresponding to $z=1$, seems to be a special case. As we describe in the following section, we find that the partial locked region disappears for $z>1$ and that, at fixed k and varying γ, the stability boundary of the locked phase corresponds to a continuous transition for $z<1$ and first order for $z>1$.

III. ANALYTICAL AND NUMERICAL RESULTS

We shall define the transition lines between different regions in the phase diagram starting from areas where the solutions are stationary. Therefore we shall search for solutions characterized by a density $p_0(\omega,\theta)$ and a velocity $v_0(\omega,\theta)$ independent of time. Clearly also r and σ are time independent in Eqs. (3) and (6).

The continuity equation (7) has stationary solutions; they satisfy $p_0v_0=C(\omega)$. From Eq. (5) we see that if $\omega<\sigma$, one has the solution $C(\omega)\equiv 0$ and therefore $v_0=0$. This implies that

$$p_0 = \delta(\theta-\theta^*), \quad \sin\,\theta^* = \frac{\omega}{\sigma}, \tag{9}$$

with the condition

$$\sigma \geqslant 1+\gamma. \tag{10}$$

The solution (9) corresponds to the state of death (all the oscillators blocked at a fixed value of θ). We shall assume $\theta^*\in(0,\pi/2)$; i.e., the result of the Winfree model [8]—as we have numerically tested that this result holds also for generic z. We shall discuss this solution in Sec. III A.

If $\omega>\sigma$, then $C(\omega)\neq 0$ and we get

$$p_0(\omega,\theta) = \frac{C(\omega)}{\omega-\sigma\sin\,\theta}. \tag{11}$$

From the normalization condition one has

$$C(\omega) = \frac{\sqrt{\omega^2-\sigma^2}}{2\pi}. \tag{12}$$

In the following we derive the stability boundaries between the phases of the model, generalizing the methods in [8,9].

A. Stability boundaries of the death phase

From Eq. (3) we get that, in general,

$$r \sin \psi = \int_0^{2\pi} d\theta \int_{1-\gamma}^{1+\gamma} d\omega g(\omega) \sin \theta p_0(\omega,\theta),$$

$$r \cos \psi = \int_0^{2\pi} d\theta \int_{1-\gamma}^{1+\gamma} d\omega g(\omega) \cos \theta p(\omega,\theta), \tag{13}$$

and in the death state

$$r \sin \psi = \frac{1}{\sigma}, \tag{14}$$

$$r \cos \psi = \int_{1-\gamma}^{1+\gamma} d\omega g(\omega) \sqrt{1-\frac{\omega^2}{\sigma^2}}, \tag{15}$$

which can be employed to determine r and ψ.

Let us use Eq. (6) to get

$$\frac{\sigma}{kr^{z-1}} = 1 + r \cos \psi \tag{16}$$

and adopt the definition

$$G_\gamma(\sigma) = \sigma r^{1-z}. \tag{17}$$

An explicit formula for the right-hand side of Eq. (16) is obtained by Eq. (15). One gets

$$1 + r\cos\psi = 1 + \frac{\sigma}{4\gamma}\left[\frac{1+\gamma}{\sigma}\sqrt{1-\left(\frac{1+\gamma}{\sigma}\right)^2} - \frac{1-\gamma}{\sigma}\sqrt{1-\left(\frac{1-\gamma}{\sigma}\right)^2} + \arcsin\frac{1+\gamma}{\sigma} - \arcsin\frac{1-\gamma}{\sigma}\right] \equiv F_\gamma(\sigma). \tag{18}$$

The properties of $F_\gamma(\sigma)$ were studied in [9]. For completeness we report here these results. It turns out that, as a function of σ and for fixed γ, $F_\gamma(\sigma)$ is a non-negative, increasing function, having concavity down. From Eq. (14) we also have r as a function of γ and σ,

$$r = \sqrt{\frac{1}{\sigma^2} + [F_\gamma(\sigma) - 1]^2}. \tag{19}$$

We distinguish the cases of large and small γ, the two ranges being separated by a limiting value $\gamma_d \in (0,1)$ that depends on z. For $z=1$, $\gamma_d=0.2956$ [9]; let us generalize this result using a procedure similar to that of [9]. At the same time we will characterize the two regions $\gamma < \gamma_d$ and $\gamma > \gamma_d$.

For a fixed value of γ such that $\gamma < \gamma_d$, the two functions $G_\gamma(\sigma)/k$ and $F_\gamma(\sigma)$ can have one, two, or no intersection, depending on the value of k. The value of σ where the two curves are tangent—i.e., $\sigma_d(\gamma)$—satisfies $\sigma_d(\gamma) > 1+\gamma$ and is the smallest value of σ such that Eq. (16) is satisfied. Therefore it characterizes the boundary. We can use Eq. (16) and the tangency condition

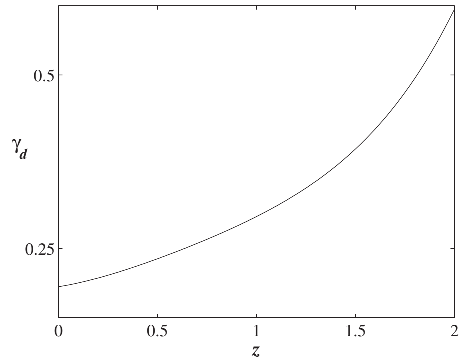

FIG. 1. The limiting value γ_d as a function of z. The Winfree model is obtained for $z=1$.

$$\frac{G'_\gamma(\sigma)}{k} = F'_\gamma(\sigma) \tag{20}$$

to extract $\sigma_d(\gamma)$, getting rid of k. In this way one gets the boundary in the form

$$k = \frac{G_\gamma(\sigma_d(\gamma))}{F_\gamma(\sigma_d(\gamma))} \quad (\gamma < \gamma_d). \tag{21}$$

The procedure can be repeated for various values of the parameter z characterizing the generalized Winfree model.

Increasing γ, σ_d decreases and eventually it reaches the value $1+\gamma$. For any γ such that $\gamma > \gamma_d$ the values k are obtained by

$$k = \frac{G_\gamma(1+\gamma)}{F_\gamma(1+\gamma)} \quad (\gamma > \gamma_d). \tag{22}$$

The limiting value γ_d is the solution of the equation

$$\frac{G_\gamma(1+\gamma)}{F_\gamma(1+\gamma)} = \frac{G'_\gamma(1+\gamma)}{F'_\gamma(1+\gamma)}. \tag{23}$$

The result, for various values of z in the interval (0,2), is reported in Fig. 1.

The separation lines between the death region and the other phases (partial death, incoherence, and locking-partial locking) are reported in Fig. 2 for four values of z: 0.5, 1, 1.5, and 2. As stated above the analytical results are confirmed by the numerical analysis. It can be noted that the boundaries of the death phase are almost independent of z. This follows from the fact that the numerical values of r are quite close to unity, as can be seen expanding in the variable γ:

$$r(\gamma,\sigma) = 1 + \frac{\gamma^2}{6(1-\sigma^2)} + \frac{(31+9\sigma^2)\gamma^4}{360(1-\sigma^2)^3} + O(\gamma^6). \tag{24}$$

B. Transition incoherence and partial death

Let us approach the boundary between these two phases from the incoherence side. We use Eq. (11) in Eq. (6). Since one must have $\sigma \leqslant 1-\gamma$, the boundary is obtained putting $\sigma = 1-\gamma$. One has

$$r \cos \psi = 0,$$

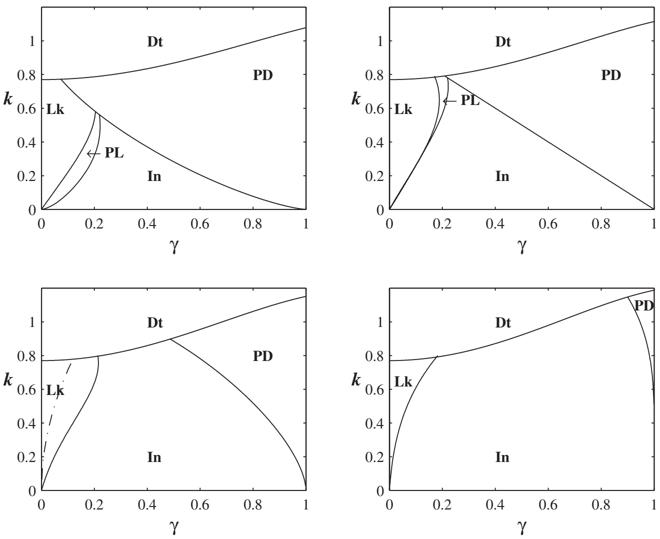

FIG. 2. Phase diagram of the generalized Winfree model. Results are for $z=0.5$ (top left), $z=1$ (top right), $z=1.5$ (bottom left), and $z=2$ (bottom right). Dt=death phase, PD=partial death, In=incoherence, Lk=locked phase, and PL=partial locking. The phase diagram at $z=1$ coincides with that depicted in [8]. In the case $z=1.5$ the stability boundary between partial locking and incoherence has been drawn as the dash-dotted line because it is not observed in simulations (for $z>1$ the partial locking region is absent; see the text).

$$r \sin \psi = \int_{1-\gamma}^{1+\gamma} d\omega g(\omega) C(\omega) \int_0^{2\pi} d\theta \frac{\sin \theta}{\omega - \sigma \sin \theta}, \quad (25)$$

so that

$$r(\gamma,\sigma) = \frac{\sigma}{2\gamma}\left[f\left(\frac{1+\gamma}{\sigma}\right) - f\left(\frac{1-\gamma}{\sigma}\right)\right], \quad (26)$$

where

$$f(x) = \frac{x^2}{2} - \frac{x\sqrt{x^2-1}}{2} + \frac{1}{2}\ln(x+\sqrt{x^2-1}). \quad (27)$$

It follows that the boundary between the two regions is given by

$$k = (1-\gamma)[r(\gamma, 1-\gamma)]^{1-z}, \quad (28)$$

with r given by Eq. (26). Also these results are reported in Fig. 2. We note that increasing z the incoherent region increases while the partial death region decreases.

C. Transition incoherence and partial locking

In order to determine the transition line we generalize the results of [9] to the case $z \neq 1$. One adds to the static solution given in Eq. (11), $p_0(\omega,\theta)=C(\omega)/(\omega-\sigma_0 \sin \theta)$, a small time-dependent perturbation

$$p(\omega,\theta) = p_0(\omega,\theta) + \epsilon \eta(\omega,\theta,t), \quad (29)$$

with $\epsilon=0^+$ and

$$\int_0^{2\pi} d\theta \eta(\omega,\theta,t) = 0. \quad (30)$$

Similarly

$$\sigma = \sigma_0 + \epsilon \sigma_1, \quad (31)$$

with

$$\sigma_0 = k r^{z-1} \quad (32)$$

and

$$\sigma_1(t) = k r^{z-1} \int d\omega g(\omega) \int_0^{2\pi} d\theta \eta(\omega,\theta,t), \quad (33)$$

so that

$$v = v_0 - \epsilon \sigma_1(t) \sin \theta, \quad (34)$$

with $v_0=\omega-\sigma_0 \sin \theta$. From the continuity equation one has, at first order in ϵ,

$$\frac{\partial \eta}{\partial t} + \frac{\partial(\eta v_0)}{\partial \theta} = \sigma_1(t)\frac{\omega C(\omega)\cos \theta}{v_0^2}. \quad (35)$$

Searching for solutions in the form

$$\eta = e^{\lambda t} h(\omega,\theta), \quad (36)$$

one finds

$$\lambda h + \frac{\partial(h v_0)}{\partial \theta} = A\frac{\omega C(\omega)\cos \theta}{v_0^2}, \quad (37)$$

with

$$A = k r^{z-1} \int d\omega g(\omega) \int_0^{2\pi} d\theta h(\omega,\theta), \quad (38)$$

whose solution is [9]

$$h(\omega,\theta) = \frac{1}{v_0^2}(a + b\cos \theta + c \sin \theta), \quad (39)$$

with

$$a = -A\frac{\sigma_0 \omega C(\omega)}{\lambda^2 + \omega^2 - \sigma_0^2}, \quad b = -\frac{\lambda a}{\sigma_0}, \quad c = -\frac{\omega a}{\sigma_0}. \quad (40)$$

Therefore from Eq. (38) one gets

$$\sigma_0^2 = \sigma_0 I_{\sigma_0}, \quad (41)$$

$$\sigma_0 = k r^{z-1}, \quad (42)$$

with

$$I_{\sigma_0} = \int d\omega g(\omega)\omega\lambda \frac{\omega^2 - \sqrt{\omega^2-\sigma_0^2}}{\lambda^2+\omega^2-\sigma_0^2} \quad (43)$$

and, as in the previous equation (26),

$$r = \frac{\sigma_0}{2\gamma}\left[f\left(\frac{1+\gamma}{\sigma_0}\right) - f\left(\frac{1-\gamma}{\sigma_0}\right)\right]. \quad (44)$$

The transition line is obtained by taking the limit $\mathrm{Re}(\lambda) \to 0^+$. Approximate results can be obtained performing a perturbative expansion for small γ, taking into account that in this limit also $\sigma_0 \to 0$. At the lowest order in γ we can write $\lambda = i + \lambda_1$ with λ_1 real. A better approximation can be obtained going up to fifth order. The result is identical to the one found in [8,9] for $z=1$, with the substitution $k \to \sigma_0$. At the lowest order in γ one gets $r = \frac{\sigma_0}{2}$ and therefore from Eq. (42)

$$\sigma_0 = 2\left(\frac{k}{2}\right)^{1/(2-z)}, \tag{45}$$

which shows that there are no solutions for $z > 2$. From Eq. (43) one finds

$$\sigma_0 = \frac{8\gamma}{\pi}\left(1 + \frac{16\gamma^2}{\pi^2} + \frac{16(\pi^2+80)\gamma^4}{\pi^4}\right) + O(\gamma^7) \tag{46}$$

and k is given by

$$k = 2\left(\frac{\sigma_0}{2}\right)^{2-z}. \tag{47}$$

In Fig. 2 the separation line between *incoherence* and *partial locking* is computed using the exact expression of Eqs. (41)–(43). The approximate formula based on the expansion (46) is valid within 4% for values of γ not larger than $\simeq 0.21$ and $z=1$; for $z=0.5$, the validity is within 4% for $\gamma < 0.19$. It should be noted that for $z > 1$ moving from the right to the left at fixed k in the diagrams, one encounters the *locking* region before reaching the *partial locking* phase. Therefore, for $z > 1$ the partial locking region is basically absent, which means that, starting from a random initial configuration, the system never reaches a partial locking state. We remark that the occurrence of this phenomenon does not depend on the choice of $g(\omega)$ uniform; indeed, we verify that it holds also for a Lorentzian distribution. We refer the reader to [19] for a discussion about the role of the shape of the distribution of frequencies in the Kuramoto model.

D. Transitions locking and partial locking ($z < 1$) and locking and incoherence ($z > 1$)

To derive the boundary from the locking to the partial locking phase, one should define the latter. This characterization can be only heuristic, given the composite nature of the partial locked phase. We have obtained the transition line by the numerical solution of Eq. (4), using $N=800$ oscillators and $T=1000$ time units. To study the stability of the locked phase one should distinguish two cases $z < 1$ and $z > 1$. In the case $z < 1$ we find a continuous transition from the locked phase towards the partial locking phase in which the time average of the order parameter decreases continuously between two limit values (see Fig. 3). For $z > 1$ there is a first-order transition from the locked phase to the incoherent phase as the time average of the order parameter r jumps from $r \sim 1$ (locked phase) to $r < 0.1$ (incoherent phase), as shown in Fig. 3. In order to derive the curves limiting the locked phase, drawn in Fig. 2, we fix the coupling constant k; for each value we consider the plot of r versus γ, and (i) for $z > 1$ find the value of γ at which the jump occurs and (ii) for $z < 1$ find the value of γ at which the curve has a flex point. The boundary line of the locked phase slightly depends on z (in the locked phase $r \approx 1$): as z decreases the locked phase region is slightly enlarged.

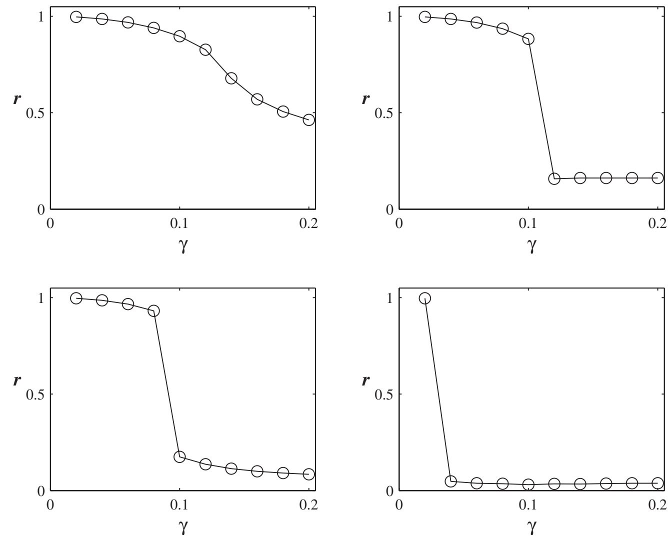

FIG. 3. The order parameter (time-averaged) r versus γ, for low γ and fixed $k=0.4$, is reported for $z=0.5$ (top left, continuous transition between the locked phase and partial locked phase), $z=1$ (top right), $z=1.5$ (bottom left), and $z=2$ (bottom right). In the last three cases there is a first-order transition between the locked phase and incoherence phase.

IV. CONCLUSIONS

We have presented a modification of the Winfree model to account for effective changes in the coupling constant among oscillators, as suggested by experiments on Josephson junctions and mechanical systems. The modification can be parametrized by a real number z. The case $z=1$ corresponds to the Winfree model; $z < 1$ leads to a coupling which decreases as the order parameter r increases, thus enforcing the coupling at low r (low amount of synchronization). At $z > 1$ the coupling increases with r; i.e., the coupling is weakened at low r. Using both an analytical approach and numerical simulations we have outlined the phase diagram of the model as z varies.

As Fig. 2 clearly shows, the death phase region is almost independent of z, while the region of incoherence is strongly influenced by this parameter: for $z < 1$ it shrinks, as the effective coupling is strengthened at a low amount of synchronization, whereas it widens at $z > 1$ at the expenses of the partial death region.

As far as the partial locking phase is concerned, we find that it disappears at $z > 1$, thus leading to the following phenomenon. At low k, as γ is increased the system leaves the locking phase through a continuous transition (in the order parameter r) for $z < 1$, while for $z > 1$ the system undergoes a discontinuous transition while leaving the locked phase. This happens because for $z < 1$ the partial locking phase separates the locking and the incoherence phases, whereas for $z > 1$ the transition is directly onto the incoherence phase. The standard case $z=1$, hence, appears to be rather special. It is worth noting that a similar change in the nature of the transition was noticed in the generalized Kuramoto model [17], and a discontinuous transition was experimentally seen in the synchronization of over damped Josephson junctions [15], where physically the parameter z corresponds to the degree of feedback provided by a coupling resonator. Our results suggest strategies to control the incoherent behavior in systems of interacting oscillators with coupling depending on the fraction of synchronized subunits.

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